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Dynamics of cantilevered pipes conveying fluid. Part 1: Nonlinear equations of three-dimensional motion

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Abstract

In a three-part study, the first part being this paper, the investigation of the three-dimensional nonlinear dynamics of unrestrained and restrained cantilevered pipes conveying fluid is undertaken. The full derivation of the equations of motion in three dimensions for the plain cantilevered pipe is presented first in this paper, using a modified version of Hamilton's principle, adapted for an open system. Intermediate (between the clamped and free end) nonlinear spring constraints are then incorporated into the equations of motion via the method of virtual work. Furthermore, a point mass fixed at the free end of the pipe is also added to the system. The equations of motion are presented in dimensionless form and then discretized with Galerkin's method.

Keywords: Pipe conveying fluid; Cantilevered; Nonlinear equations; 3D motions; Additional intraspan spring support; Additional end-mass; Hamilton's principle

1. Introduction

Although the dynamics of pipes conveying fluid has been studied extensively over the past 50 years or so, most of the theoretical models utilized have been two-dimensional.

Some of the earliest work [e.g. Feodos'ev (1951), Housner (1952)] showed that pinned–pinned and clamped–clamped pipes conveying fluid may lose stability by divergence, i.e. they may buckle, at sufficiently high flow velocities. Benjamin (1961a, b) was the first to undertake a thorough study of the dynamics of a cantilevered pipe conveying fluid; this pioneering work is particularly well known for formulating a modified version of Hamilton's principle accounting for momentum flux in and out of the open system, which is still used today and will in fact be used in this work. Later work by Gregory and Païdoussis (1966a, b) showed that cantilevered pipes lose stability via flutter for sufficiently high flow velocities. Two-dimensional (2-D) theory is sufficient for predicting the buckling (static divergence) of a simply supported pipe, or the dynamic instability of a cantilevered one, since these are physically planar bifurcations.

In 1970–2000 period, the literature on this topic literally exploded, with linear and nonlinear, two- and three-dimensional, theoretical and experimental studies, both for the plain pipe and for pipes with added spring supports, masses and dashpots, somewhere along the length of the pipe. This was the result of gradual realization that this

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problem is a new paradigm in dynamics, displaying in a simple system a wide variety of the dynamics of more complex engineering problems; moreover, the fluid-conveying pipe system has the advantage that one can fairly easily conduct experiments to test the theory. A thorough literature review will not be undertaken in this paper. For that, the reader is referred to Païdoussis and Li (1993) and Païdoussis (1998). Also, more complete literature reviews on the particular topics of the Parts 2 and 3 papers may be found therein.

As is evident from even a casual perusal of Païdoussis (1998), the cantilevered pipe conveying fluid, being an inherently nonconservative system, is capable of displaying a seemingly inexhaustible variety of interesting and often surprising behaviour—much more than a pipe with both ends supported. Accordingly, the system considered in this paper, whether with intermediate springs or with an end mass (defined below), involves a cantilevered pipe.

The purpose of this paper is to present the derivation of the three-dimensional (3-D) nonlinear equations of motion of the fluid-conveying cantilevered pipe, also incorporating an ‘‘intermediate’’ or ‘‘intraspan’’ spring support (i.e. a spring support somewhere between the fixed and the free end of the pipe), as well as an ‘‘end-mass’’ (i.e. a relatively small mass attached at the free end of the pipe). Remarkable previous work on the dynamics of the plain cantilevered pipe has been conducted by Bajaj and Sethna (1984), and for the pipe fitted with a bent end-nozzle by Lundgren et al. (1979). Equally remarkable is a series of studies by Steindl and Troger (1988, 1991, 1995, 1996), focussing on the 3-D motions of a cantilevered pipe with an intermediate spring support, revealing a veritable cornucopia of dynamical behaviour. In these studies, different, but equivalent, forms of the 3-D equations of motion to those obtained here were derived.

The equations derived in this (Part 1) paper are used in Part 2 (Païdoussis et al., 2007), which deals exclusively with the 3-D nonlinear dynamics of a cantilevered pipe with an intermediate spring support. In addition to the aforementioned work by Steindl and Troger, the theoretical and experimental work by Païdoussis and Semler (1993) should be mentioned. The main objective of that paper was the study of the dynamics in the neighbourhood of the state of double degeneracy, when buckling (divergence) and flutter occur at the same flow velocity, achieved by a judicious choice of location and stiffness of the intermediate spring.

The dynamics of a cantilevered pipe with an end-mass is the subject of the Part 3 paper (Modarres-Sadeghi et al., 2007). In this case also, a literature review may be found in that paper. However, the work by Païdoussis and Semler (1998) should be mentioned, and particularly that of Copeland and Moon (1992); in the latter, it was shown that, depending on the size and weight of the end-mass, as well as the flow velocity, an intricate sequence of generally 3-D and chaotic oscillations can occur.

In both Parts 2 and 3, the theoretical results, obtained by means of the equations derived here, are compared with experimental observations and previous theoretical models. Indeed, it should be said that the rich 3-D dynamics observed in the aforementioned experiments, mainly those by Copeland and Moon (1992) but also those by Païdoussis and Semler (1993, 1998), was responsible for giving the main impetus for this three-part study.

In Parts 2 and 3 of this study, the main interest is in the post-critical dynamics of the system; i.e. the dynamics beyond the first bifurcation encountered by the system, which can also be predicted by linear theory. For the system with an intraspan spring (Part 2), this first bifurcation is related to loss of stability by divergence or flutter, depending mainly on the location and strength of these springs. In Part 2, the dynamics beyond that point is explored (i.e. for higher flow rates), focussing on the determination and characterization of any additional bifurcations. The same applies to the system with an added end mass (Part 3), where the dynamics is explored beyond the onset of flutter, focussing on additional bifurcations and on whether the motions are two- or three-dimensional.

Finally, it should be made clear that this three-part paper reports on a fundamental, curiosity-driven investigation. Although a number of applications do exist for the general topic of pipes conveying fluid (Païdoussis, 1998, Section 4.7), e.g. the Coriolis mass-flow meter, ocean mining and the hydroelastic ichthyoid propulsion system, the main motivation for this work does not come from applications. Nevertheless, as amply demonstrated in Païdoussis (1993), although research in the broad area of fluid–structure interactions involving axial flow is often undertaken with little or no application in mind, unexpected uses and applications emerge, 10, 20 or 50 years later.

2. Problem statement

In this paper, we consider a vertical cantilevered pipe of length L, mass per unit length m and of flexural rigidity EI, conveying a fluid of mass per unit length M with mean axial velocity U; see Fig. 1.

The equations of motion are obtained under the following assumptions: (i) the fluid is incompressible; (ii) the flow is of constant velocity and free from pulsation; (iii) the pipe behaves like a nonlinear Euler–Bernoulli beam (the diameter is small compared to the length); (iv) the strain in the pipe is considered small, although large deflections are expected; (v) rotary inertia and shear deformation are neglected; (vi) the pipe centreline is inextensible; (vii) in the case of the pipe

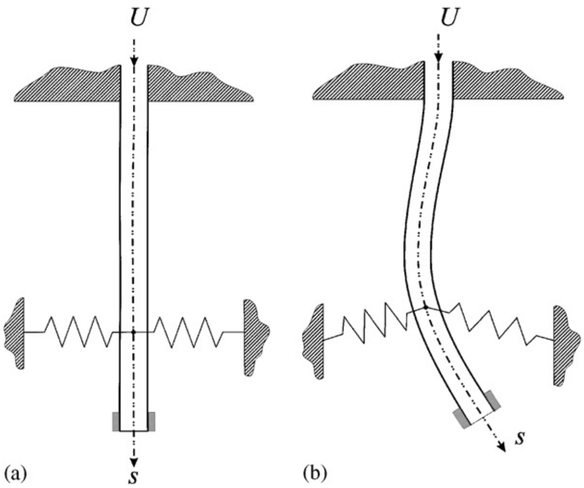

Fig. 1. Diagram of a cantilevered pipe conveying fluid with flow velocity U. Also shown are "intermediate springs" (of individual stiffness k) and an end-mass (of mass m_e) which may be added to the basic system. (a) Undeformed system; (b) deformed system.

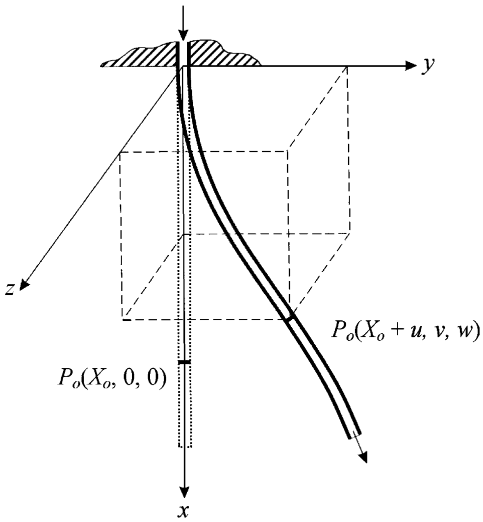

Fig. 2. Diagram illustrating the Lagrangian (X_o, Y_o, Z_o) and Eulerian (x, y, z) coordinate systems and a representative segment on the pipe centreline identified according to the Lagrangian system, $P_o(X_o, 0, 0)$, and the Eulerian system, $P_o(x, y, z) \equiv P_o(X_o + u, v, w)$.

with an intermediate spring support, the springs are assumed to be attached to the centreline of the pipe[1]; (viii) the end-mass is assumed to be a point mass.

The Lagrangian coordinate system is introduced here as (X_0, Y_0, Z_0), labelling specific particles at a certain place and time taken at the original equilibrium state of the pipe, to create a relation between the Eulerian coordinate system (x, y, z) and the displacement (u, v, w) of any particular material point on the pipe (Fig. 2). The two coordinate systems are related as follows:

$$x = X_0 + u, \quad y = Y_0 + v, \quad z = Z_0 + w. \tag{1}$$

[1]This assumption will be modified in Appendix C of this paper and will be discussed in Part 2 (Païdoussis et al., 2007) of this study.

By setting the Eulerian coordinate x, or the Lagrangian coordinate X_0, along the pipe centreline in the equilibrium configuration when the free pipe is at rest, Y_0 and Z_0 become zero. The x-axis is defined to coincide with the direction of gravity.

It is useful to define a curvilinear coordinate s, along the length of the pipe. Because the pipe has been assumed to be inextensible, it follows that $X_0 \equiv s$. Through simple derivations (Païdoussis, 1998) the inextensibility condition may be stated as follows:

$$\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 + \left(\frac{\partial z}{\partial s}\right)^2 = 1, \tag{2}$$

or

$$\left(1 + \frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial v}{\partial s}\right)^2 + \left(\frac{\partial w}{\partial s}\right)^2 = 1. \tag{3}$$

The inextensibility condition will be used, in different variations of the above and often as a linear or third-order approximation, to reduce the system of three partial differential equations of motion down to two, involving y and z, or in terms of v and w.

The methods used to obtain the 3-D model presented here were inspired by previous derivations of the 2-D governing equation of motion for a cantilevered pipe conveying fluid (Semler et al., 1994).

The starting point of the derivation is a modified version of Hamilton's principle, where the right-hand side accounts for the energy gained or lost at the free end of the pipe (Benjamin, 1961a,b):

$$\delta \int_{t_1}^{t_2} \mathscr{L}\, \mathrm{d}t + \int_{t_1}^{t_2} \delta W\, \mathrm{d}t = \int_{t_1}^{t_2} \left\{ MU \left[\frac{\partial \boldsymbol{r}_L}{\partial t} + U \boldsymbol{\tau}_L \right] \cdot \delta \boldsymbol{r}_L \right\} \mathrm{d}t. \tag{4}$$

In Eq. (4), $\mathscr{L}$ represents the Lagrangian of the system ($\mathscr{L} = T_p + T_f - V_p - V_f$), where T and V represent the kinetic and potential energy, respectively, and the subscripts p and f refer to the pipe and the fluid, respectively; δW is the virtual work which accounts for the forces not included in the Lagrangian. The right-hand side of Eq. (4) can be viewed as the virtual momentum transport across the open surface at the end of the pipe. The subscript L on the right-hand side of Eq. (4) specifies that the vector is measured at $s = L$.

On the right-hand side of Eq. (4), $\boldsymbol{r}$ is a position vector and $\boldsymbol{\tau}$ is the tangent vector at any point of the pipe. They may be expressed as follows:

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = (s + u)\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}, \tag{5}$$

$$\boldsymbol{\tau} = \frac{\partial x}{\partial s}\boldsymbol{i} + \frac{\partial y}{\partial s}\boldsymbol{j} + \frac{\partial z}{\partial s}\boldsymbol{k} = \left(1 + \frac{\partial u}{\partial s}\right)\boldsymbol{i} + \frac{\partial v}{\partial s}\boldsymbol{j} + \frac{\partial w}{\partial s}\boldsymbol{k}. \tag{6}$$

Although assumption (iv) states that one can expect large deflections, an order of magnitude analysis in the planes of lateral motion shows that z (or displacement w) and y (or displacement v) can be considered small, defined as being of the order ε, such that

$$z = w = \mathcal{O}(\varepsilon), \tag{7}$$

$$y = v = \mathcal{O}(\varepsilon). \tag{8}$$

The derivation of the equations of motion undertaken here accounts only for nonlinear terms up to the third order, $\mathcal{O}(\varepsilon^3)$.[2]

Additional constraints to the system, such as an intermediate spring support, can be incorporated into the equations of motion by the principle of virtual work. In Eq. (4) the virtual work, δW, represents the work done by the sum of forces acting on the pipe, as a result of a virtual displacement.

Let us consider a distributed force (moment) along the length of the pipe, $Q(s)$, which is a function of the centreline deformation, $u(s,t)$, $v(s,t)$ and $w(s,t)$ (or corresponding rotations if Q is a moment), i.e.

$$Q = Q(u, v, w) = Q_1\boldsymbol{i} + Q_2\boldsymbol{j} + Q_3\boldsymbol{k}, \tag{9}$$

[2] In the derivations, $\{x, y, z\}$ or $\{u, v, w\}$ are used, as convenient. Particularly in all derivations related to the intermediate springs the $\{u, v, w\}$ system is used. The final equations of motion will be expressed in terms of u, v and w (u will be implicitly present through the inextensibility condition).

which may be rewritten such that

$$Q_1 = F_u, \quad Q_2 = F_v, \;\; Q_3 = F_w, \tag{10}$$

in which F_u, F_v and F_w may be thought of as forces per unit length acting, respectively, in the x, y and z directions. The virtual work done by these forces, associated with displacements δu, δv and δw, is

$$\delta W = \int_0^L (F_u \delta u + F_v \delta v + F_w \delta w)\,\mathrm{d}s. \tag{11}$$

One more relationship that will come handy is the expression for curvature in three-dimensional coordinates (Lundgren et al., 1979), expressed as

$$\kappa^2 = \left(\frac{\partial^2 x}{\partial s^2}\right)^2 + \left(\frac{\partial^2 y}{\partial s^2}\right)^2 + \left(\frac{\partial^2 z}{\partial s^2}\right)^2 = \left(\frac{\partial^2 u}{\partial s^2}\right)^2 + \left(\frac{\partial^2 v}{\partial s^2}\right)^2 + \left(\frac{\partial^2 w}{\partial s^2}\right)^2. \tag{12}$$

3. The plain cantilevered pipe

The kinetic energy of a plain cantilevered pipe (i.e. in the absence of spring constraints and end-mass) conveying fluid can be expressed as follows:

$$T = T_p + T_f = \frac{1}{2} m \int_0^L \boldsymbol{v}_p^2 \,\mathrm{d}s + \frac{1}{2} M \int_0^L \boldsymbol{v}_f^2 \,\mathrm{d}s, \tag{13}$$

where $\boldsymbol{v}_p$ and $\boldsymbol{v}_f$ are the pipe and fluid velocities, defined, respectively, by

$$\boldsymbol{v}_p = \frac{\partial \boldsymbol{r}}{\partial t} = \frac{\partial x}{\partial t}\boldsymbol{i} + \frac{\partial y}{\partial t}\boldsymbol{j} + \frac{\partial z}{\partial t}\boldsymbol{k} = \frac{\partial u}{\partial t}\boldsymbol{i} + \frac{\partial v}{\partial t}\boldsymbol{j} + \frac{\partial w}{\partial t}\boldsymbol{k}, \tag{14}$$

$$\boldsymbol{v}_f = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial s}\right)(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \equiv \frac{\mathrm{D}\boldsymbol{r}}{\mathrm{D}t}. \tag{15}$$

Substituting Eqs. (14) and (15) into Eq. (13), then applying Hamilton's variational operator, δ, on the latter, and using the inextensibility condition, one obtains after some manipulation

$$\begin{aligned}
\delta \int_{t_1}^{t_2} T\,\mathrm{d}t = & -\int_{t_1}^{t_2}\int_0^L (m+M)\left[\ddot{y} + y'\int_0^s (\dot{y}'^2 + y'\ddot{y}' + \dot{z}'^2 + z'\ddot{z}')\,\mathrm{d}s\right]\delta y\,\mathrm{d}s\,\mathrm{d}t \\
& + \int_{t_1}^{t_2}\int_0^L (m+M)\left[y''\int_0^s\int_s^L (\dot{y}'^2 + y'\ddot{y}' + \dot{z}'^2 + z'\ddot{z}')\,\mathrm{d}s\,\mathrm{d}s\right]\delta y\,\mathrm{d}s\,\mathrm{d}t \\
& - \int_{t_1}^{t_2}\int_0^L 2MU\left[\dot{y}'(1+y'^2) + y'z'\dot{z}' - y''\int_s^L (y'\dot{y}' + z'\dot{z}')\,\mathrm{d}s\right]\delta y\,\mathrm{d}s\,\mathrm{d}t \\
& - \int_{t_1}^{t_2}\int_0^L (m+M)\left[\ddot{z} + z'\int_0^s (\dot{z}'^2 + z'\ddot{z}' + \dot{y}'^2 + y'\ddot{y}')\,\mathrm{d}s\right]\delta z\,\mathrm{d}s\,\mathrm{d}t \\
& + \int_{t_1}^{t_2}\int_0^L (m+M)\left[z''\int_0^s\int_s^L (\dot{z}'^2 + z'\ddot{z}' + \dot{y}'^2 + y'\ddot{y}')\,\mathrm{d}s\,\mathrm{d}s\right]\delta z\,\mathrm{d}s\,\mathrm{d}t \\
& - \int_{t_1}^{t_2}\int_0^L 2MU\left[\dot{z}'(1+z'^2) + z'y'\dot{y}' - z''\int_s^L (z'\dot{z}' + y'\dot{y}')\,\mathrm{d}s\right]\delta z\,\mathrm{d}s\,\mathrm{d}t \\
& + MU\int_{t_1}^{t_2} (\dot{x}_L\delta x_L + \dot{y}_L\delta y_L + \dot{z}_L\delta z_L)\,\mathrm{d}t,
\end{aligned} \tag{16}$$

where the overdot stands for $\partial(\;)/\partial t$, and the prime stands for $\partial(\;)/\partial s$.

The potential energy of the system has two components: the pipe strain energy, V_{ps}, and the gravitational energy of both pipe and fluid, $V_g = V_{pg} + V_{fg}$. The expression for potential strain energy, related uniquely to the material properties of the pipe, may be written as

$$\delta \int_{t_1}^{t_2} V_{ps}\,\mathrm{d}t = \frac{1}{2}EI \int_{t_1}^{t_2} \int_0^L \delta(\kappa^2)\,\mathrm{d}s\,\mathrm{d}t = \frac{1}{2}EI \int_{t_1}^{t_2} \int_0^L (\delta x''^2 + \delta y''^2 + \delta z''^2)\,\mathrm{d}s\,\mathrm{d}t, \tag{17}$$

where κ is the curvature of the pipe.

As seen above, the variational operator needs to be applied to the expression for curvature, Eq. (12). The inextensibility condition, Eq. (2), is then applied to the x-component of curvature, leading to the following expression for the strain energy in the system:

$$\begin{aligned}
\delta \int_{t_1}^{t_2} V_{ps}\,\mathrm{d}t &= \frac{1}{2}EI \int_{t_1}^{t_2} \int_0^L (\delta(y'^2 y''^2 + 2y'y''z'z'' + z'^2 z''^2) + \delta y''^2 + \delta z''^2)\,\mathrm{d}s\,\mathrm{d}t \\
&= EI \int_{t_1}^{t_2} \int_0^L (y'''' + y''^3 + 4y'y''y''' + y'^2 y'''')\delta y\,\mathrm{d}s\,\mathrm{d}t \\
&+ EI \int_{t_1}^{t_2} \int_0^L (y''z''^2 + y''z'z''' + 3y'z''z''' + y'z'z'''')\delta y\,\mathrm{d}s\,\mathrm{d}t \\
&+ EI \int_{t_1}^{t_2} \int_0^L (z''''z''^3 + 4z'z''z''' + z'^2 z'''')\delta z\,\mathrm{d}s\,\mathrm{d}t \\
&+ EI \int_{t_1}^{t_2} \int_0^L (z''y''^2 + z''y'y''' + 3z'y''y''' + z'y'y'''')\delta y\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned} \tag{18}$$

The potential energy due to gravity can be expressed as

$$V_g = -(m+M)g \int_0^L x\,\mathrm{d}s. \tag{19}$$

By substituting the inextensibility condition into Eq. (19), applying the variational operator and integrating by parts, the following expression may be obtained:

$$\begin{aligned}
\delta \int_{t_1}^{t_2} V_g\,\mathrm{d}t =& -(m+M)g \int_{t_1}^{t_2} \int_0^L \delta x\,\mathrm{d}s \\
=& (m+M)g \int_{t_1}^{t_2} \int_0^L \left[\left(z' + \frac{1}{2}z'^3 + \frac{1}{2}z'y'^2\right)\right]\delta z\,\mathrm{d}s\,\mathrm{d}t \\
&- (m+M)g \int_{t_1}^{t_2} \int_0^L \left[(L-s)\left(z'' + \frac{3}{2}z'^2 z'' + \frac{1}{2}y'^2 z'' + z'y'y''\right)\right]\delta z\,\mathrm{d}s\,\mathrm{d}t \\
&+ (m+M)g \int_{t_1}^{t_2} \int_0^L \left[\left(y' + \frac{1}{2}y'^3 + \frac{1}{2}y'z'^2\right)\right]\delta y\,\mathrm{d}s\,\mathrm{d}t \\
&- (m+M)g \int_{t_1}^{t_2} \int_0^L \left[(L-s)\left(y'' + \frac{3}{2}y'^2 y'' + \frac{1}{2}z'^2 y'' + y'z'z''\right)\right]\delta y\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned} \tag{20}$$

Finally, the terms associated with end effects due to the open nature of the system are developed by substituting Eqs. (5) and (6) into the right-hand side of Eq. (4), yielding

$$\begin{aligned}
&\int_{t_1}^{t_2} \left\{ MU\left[\frac{\partial \boldsymbol{r}_L}{\partial t} + U\boldsymbol{\tau}_L\right] \cdot \delta \boldsymbol{r}_L \right\} \mathrm{d}t \\
&= \int_{t_1}^{t_2} MU[\dot{x}_L\boldsymbol{i} + \dot{y}_L\boldsymbol{j} + \dot{z}_L\boldsymbol{k} + U(x'_L\boldsymbol{i} + y'_L\boldsymbol{j} + z'_L\boldsymbol{k})] \cdot (\delta x_L\boldsymbol{i} + \delta y_L\boldsymbol{j} + \delta z_L\boldsymbol{k})\,\mathrm{d}t \\
&= MU \int_{t_1}^{t_2} (\dot{x}_L\delta x_L + \dot{y}_L\delta y_L + \dot{z}_L\delta z_L)\,\mathrm{d}t + MU^2 \int_{t_1}^{t_2} (x'_L\delta x_L + y'_L\delta y_L + z'_L\delta z_L)\,\mathrm{d}t.
\end{aligned} \tag{21}$$

It should be noticed that the first part of Eq. (21) will cancel out with the last part of the kinetic energy expression, Eq. (16). The last part of Eq. (21) must undergo certain manipulations; to this end, it may be shown that

$$\begin{aligned} MU^2 \int_{t_1}^{t_2} (x'_L \delta x_L + y'_L \delta y_L + z'_L \delta z_L)\, \mathrm{d}t \\ = MU^2 \int_{t_1}^{t_2} \int_0^L (x'' \delta x + x' \delta x' + y'' \delta y + y' \delta y' + z'' \delta z + z' \delta z')\, \mathrm{d}s\, \mathrm{d}t. \end{aligned} \tag{22}$$

The inextensibility condition can be rewritten as: $x'\delta x' + y'\delta y' + z'\delta z' = 0$, so that Eq. (22) becomes

$$MU^2 \int_{t_1}^{t_2} \int_0^L (x'' \delta x + y'' \delta y + z'' \delta z)\, \mathrm{d}s\, \mathrm{d}t. \tag{23}$$

Applying the inextensibility condition again, integrating by parts, using the following property of integrals (Semler et al., 1994):

$$\int_0^L g(s) \left(\int_0^s f(s) \delta y\, \mathrm{d}s \right) \mathrm{d}s = \int_0^L \left(\int_s^L g(s)\, \mathrm{d}s \right) f(s) \delta y\, \mathrm{d}s, \tag{24}$$

and after some further manipulation, the right-hand side of the modified form of Hamilton's principle becomes

$$\begin{aligned} MU^2 \int_{t_1}^{t_2} (x'_L \delta x_L + y'_L \delta y_L + z'_L \delta z_L)\, \mathrm{d}t \\ = MU^2 \int_{t_1}^{t_2} \int_0^L (x'' \delta x + y'' \delta y + z'' \delta z)\, \mathrm{d}s\, \mathrm{d}t \\ = MU^2 \int_{t_1}^{t_2} \int_0^L \left(y''(1 + y'^2) - y'' \int_s^L y'y''\, \mathrm{d}s + z'z''y' - y'' \int_s^L z'z''\, \mathrm{d}s \right) \delta y\, \mathrm{d}s\, \mathrm{d}t \\ + MU^2 \int_{t_1}^{t_2} \int_0^L \left(z''(1 + z'^2) - z'' \int_s^L z'z''\, \mathrm{d}s + y'y''z' - z'' \int_s^L y'y''\, \mathrm{d}s \right) \delta z\, \mathrm{d}s\, \mathrm{d}t. \end{aligned} \tag{25}$$

Eqs. (16), (18), (20) and (25) are sufficient for obtaining the equations of motion for the plain pipe. Instead of doing so, the spring and end-mass forces are derived first; the equations of motion for the plain pipe can then be obtained by eliminating the appropriate terms from the final form of the equations, Eqs. (40) and (41).

4. The pipe with four intermediate springs

An array of four linear springs attached to the centreline of the pipe at an intermediate position $s = L_s$, and symmetrically disposed with respect to the y and z axes is considered (a schematic representation is given in Fig. 3).

The forces that an array of N identical springs exerts on the pipe can be expressed as

$$F_s = k \sum_{i=1}^{N} (R_i - L_o) \boldsymbol{n}_{R_i}, \tag{26}$$

here $N = 4$, and $i = 1, 2, 3, 4$ designates each spring of the array. The coefficient k is the linear stiffness of a spring, and L_o its unstretched length. R_i is the length of spring i when stretched, and $\boldsymbol{n}_{R_i}$ is a unit vector along its length, the direction of which is shown in Fig. 3. The scalar value of R_i is determined according to dimensions P and Q (Fig. 3) and the pipe displacements at $s = L_s$: u_{L_s}, v_{L_s} and w_{L_s}.

Although the spring coefficient is assumed to be linear, the equivalent forces of the four springs acting on the pipe become nonlinear due to the deformation of the system (geometric nonlinearity).

Unlike the derivations for the plain pipe, which can be found in detail elsewhere [e.g. in Semler et al. (1994)], this is the first presentation of the nonlinear forces associated with the intermediate springs. Therefore, some of the intermediate steps in the derivation are given in fair detail in Appendix A. As already remarked, these derivations, for

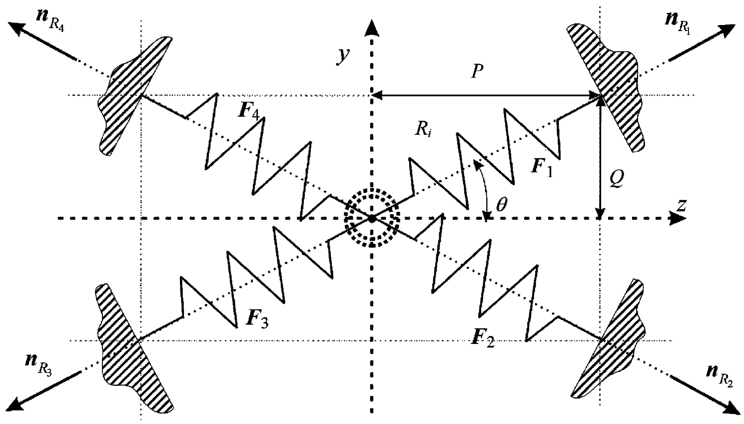

Fig. 3. Forces exerted on the pipe due to the four-spring configuration, when the springs are assumed to be connected to the centreline of the pipe. The springs are disposed in symmetrical fashion with respect to the z and y axes. θ, P and Q are predetermined and $P^2 + Q^2 = R_o^2$.

convenience, will be done in the $\{u, v, w\}$ framework, rather than using $\{x, y, z\}$. Briefly, after application of the inextensibility condition and Taylor expansions, the components of the spring forces in Eq. (26) may be written as

$$F_u = 2k\left(1 - \frac{L_o}{R_o}\right)\delta(s - L_s)\int_0^s (v'^2 + w'^2)\,\mathrm{d}s, \tag{27}$$

$$F_v = \delta(s - L_s)\left\{-4k\left(1 - \frac{L_o}{R_o}\cos^2\theta\right)v \right.$$
$$\left. -2k\frac{L_o}{R_o^3}\cos^2\theta(\cos^2\theta - 4\sin^2\theta)v^3 - 2k\frac{L_o}{R_o^3}(15\cos^2\theta\sin^2\theta - 2)vw^2\right\}, \tag{28}$$

$$F_w = \delta(s - L_s)\left\{-4k\left(1 - \frac{L_o}{R_o}\sin^2\theta\right)w \right.$$
$$\left. -2k\frac{L_o}{R_o^3}\sin^2\theta(\sin^2\theta - 4\cos^2\theta)w^3 - 2k\frac{L_o}{R_o^3}(15\cos^2\theta\sin^2\theta - 2)wv^2\right\}. \tag{29}$$

The spring forces are expressed in component form in Eqs. (27)–(29); the subscripts u, v and w are associated with the forces acting in the x, y and z directions, respectively. The Dirac delta function, $\delta(s - L_s)$, simply indicates that these equivalent forces apply only at one point along the centreline of the pipe, i.e. where the springs are assumed to be attached.

Note that θ is measured with respect to the z-axis. Thus, for $\theta < 45^\circ$ the spring array is narrower along the z-axis, as shown in Fig. 3.

For a spring array of specified geometry, expressions (27)–(29) may be rewritten in a simplified manner as

$$F_u = K_x\delta(s - L_s)\int_0^s (v'^2 + w'^2)\,\mathrm{d}s, \tag{30}$$

$$F_v = \delta(s - L_s)(-K_{yl}v - K_{ynl}v^3 - K_{yz}vw^2), \tag{31}$$

$$F_w = \delta(s - L_s)(-K_{zl}w - K_{znl}w^3 - K_{yz}wv^2), \tag{32}$$

where the K's represent constant coefficients. The subscripts make reference to the direction in which they have influence (x, y or z) as well as to whether they are associated with linear (l) or nonlinear (nl) terms.

The forces acting on the pipe may be introduced into Hamilton's principle through the principle of virtual work, as in Eq. (4). The virtual work associated with virtual displacements δu, δv and δw is given by

$$\delta W_u = \int_0^L F_u\delta u\,\mathrm{d}s = \int_0^L \left(K_x\int_0^s (v'^2 + w'^2)\,\mathrm{d}s\right)\delta(s - L_s)\,\delta u\,\mathrm{d}s, \tag{33}$$

$$\delta W_v = \int_0^L F_v \delta v \, \mathrm{d}s = -\int_0^L (K_{yl} v + K_{ynl} v^3 + K_{yz} v w^2)\delta(s - L_s)\, \delta v \, \mathrm{d}s, \tag{34}$$

$$\delta W_w = \int_0^L F_w \delta w \, \mathrm{d}s = -\int_0^L (K_{zl} w + K_{znl} w^3 + K_{yz} w v^2)\delta(s - L_s)\, \delta w \, \mathrm{d}s. \tag{35}$$

The inextensibility condition is then applied to the expression of virtual work acting in the axial-direction, δW_u, together with Eq. (24). After some further manipulation, an expression compatible with Hamilton's principle for the virtual work done on the pipe, at $s = L_s$, by an array of four intermediate springs is obtained, namely

$$\begin{aligned}
\int_{t_1}^{t_2} \delta W \, \mathrm{d}t = &\int_{t_1}^{t_2} (\delta W_u + \delta W_v + \delta W_w)\, \mathrm{d}t \\
= &-\int_{t_1}^{t_2}\int_0^L (K_{zl} w + K_{znl} w^3 + K_{yz} w v^2)\delta(s - L_s)\, \delta w \, \mathrm{d}s\, \mathrm{d}t \\
&-\int_{t_1}^{t_2}\int_0^L \left[\left(K_x \int_0^s (v'^2 + w'^2)\, \mathrm{d}s\right)\delta(s - L_s) w'\right] \delta w \, \mathrm{d}s\, \mathrm{d}t \\
&+\int_{t_1}^{t_2}\int_0^L \left[K_x w'' \mu(0 \to L_s) \int_0^{L_s} (v'^2 + w'^2)\, \mathrm{d}s\right] \delta w \, \mathrm{d}s\, \mathrm{d}t \\
&-\int_{t_1}^{t_2}\int_0^L (K_{yl} v + K_{ynl} v^3 + K_{yz} v w^2)\delta(s - L_s)\, \delta v \, \mathrm{d}s\, \mathrm{d}t \\
&-\int_{t_1}^{t_2}\int_0^L \left[\left(K_x \int_0^s (v'^2 + w'^2)\, \mathrm{d}s\right)\delta(s - L_s) v'\right] \delta v \, \mathrm{d}s\, \mathrm{d}t \\
&+\int_{t_1}^{t_2}\int_0^L \left[K_x v'' \mu(0 \to L_s) \int_0^{L_s} (v'^2 + w'^2)\, \mathrm{d}s\right] \delta v \, \mathrm{d}s\, \mathrm{d}t,
\end{aligned} \tag{36}$$

where $\mu(0 \to L_s)$ is a Heavyside function defined as having a value of 1 for the interval $s = 0$ to L_s and a value of 0 from $s = L_s$ to L.

Certain variations to the basic system of four springs as analyzed above are considered in the Part 2 paper (Païdoussis et al., 2007): (i) a two- rather than four-spring configuration, and (ii) a more realistic geometry of attachment of the springs to the pipe. Accordingly, a system with a two-spring configuration is considered in Appendix B, and the moments induced by an array of springs not attached to the centreline of the pipe but rather to a ring mounted on the pipe are found in Appendix C.

5. The pipe with an end-mass

It is quite simple to add a point mass into the variational statement of Hamilton's principle. It is assumed that a mass m_e is located at the end of the pipe; then the Lagrangian, $\mathscr{L}_e$, associated with only this mass, takes the following form:

$$\mathscr{L}_e = T_e - V_e = \tfrac{1}{2} m_e (\dot{x}_L^2 + \dot{y}_L^2 + \dot{z}_L^2) - m_e g x_L. \tag{37}$$

The steps followed to obtain the terms in the equations of motion related to the kinetic energy of the point mass are comparable to those for the kinetic energy of the plain pipe. The kinetic energy associated with the end mass, similarly to Eq. (13), may be written as

$$T_e = \frac{1}{2} m_e \int_0^L \delta(s - L)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\, \mathrm{d}s,$$

and following a comparable procedure as in the foregoing, the variational term, similar to Eq. (16), is

$$\begin{aligned}
\delta \int_{t_1}^{t_2} T_e \, \mathrm{d}t = &\frac{1}{2} m_e \delta \int_{t_1}^{t_2}\int_0^L \delta(s - L)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\, \mathrm{d}s\, \mathrm{d}t \\
= &-\int_{t_1}^{t_2}\int_0^L m_e \delta(s - L)\left[\ddot{y} + y' \int_0^s (\dot{y}'^2 + y'\ddot{y}' + \dot{z}'^2 + z'\ddot{z}')\, \mathrm{d}s\right] \delta y \, \mathrm{d}s\, \mathrm{d}t
\end{aligned}$$

$$+\int_{t_1}^{t_2}\int_0^L\left[y''\int_s^L m_e\delta(s-L)\int_0^s(\dot{y}'^2+y'\ddot{y}'+\dot{z}'^2+z'\ddot{z}')\,\mathrm{d}s\,\mathrm{d}s\right]\delta y\,\mathrm{d}s\,\mathrm{d}t$$
$$-\int_{t_1}^{t_2}\int_0^L m_e\delta(s-L)\left[\ddot{z}+z'\int_0^s(\dot{z}'^2+z'\ddot{z}'+\dot{y}'^2+y'\ddot{y}')\,\mathrm{d}s\right]\delta z\,\mathrm{d}s\,\mathrm{d}t$$
$$+\int_{t_1}^{t_2}\int_0^L\left[z''\int_s^L m_e\delta(s-L)\int_0^s(\dot{z}'^2+z'\ddot{z}'+\dot{y}'^2+y'\ddot{y}')\,\mathrm{d}s\,\mathrm{d}s\right]\delta z\,\mathrm{d}s\,\mathrm{d}t. \tag{38}$$

As a result of the point-mass assumption, there is no strain energy to account for; hence the potential energy due to the end mass is simply gravitational and it is found to be

$$\begin{aligned}\delta\int_{t_1}^{t_2}V_e\,\mathrm{d}t&=\delta\int_{t_1}^{t_2}m_e g x_L\,\mathrm{d}t=\int_{t_1}^{t_2}m_e g\delta x_L\,\mathrm{d}t\\&=-\int_{t_1}^{t_2}\int_0^L m_e g\left[\delta(s-L)\left(z'+\frac{1}{2}z'^3+\frac{1}{2}z'y'^2\right)\right]\delta z\,\mathrm{d}s\,\mathrm{d}t\\&\quad+\int_{t_1}^{t_2}\int_0^L m_e g\left[\left(z''+\frac{3}{2}z'^2z''+\frac{1}{2}y'^2z''+z'y'y''\right)\int_s^L\delta(s-L)\,\mathrm{d}s\right]\delta z\,\mathrm{d}s\,\mathrm{d}t\\&\quad-\int_{t_1}^{t_2}\int_0^L m_e g\left[\delta(s-L)\left(y'+\frac{1}{2}y'^3+\frac{1}{2}y'z'^2\right)\right]\delta y\,\mathrm{d}s\,\mathrm{d}t\\&\quad+\int_{t_1}^{t_2}\int_0^L m_e g\left[\left(y''+\frac{3}{2}y'^2y''+\frac{1}{2}z'^2y''+y'z'z''\right)\int_s^L\delta(s-L)\,\mathrm{d}s\right]\delta y\,\mathrm{d}s\,\mathrm{d}t.\end{aligned} \tag{39}$$

The kinetic and potential energies for the pipe, fluid and end-mass, the end effect due to the fluid exiting the pipe as well as the work done by the array of springs can all be combined together by substituting expressions (16), (18), (20), (25), (36), (38) and (39) into Hamilton's principle, Eq. (4). This generates one large equation involving a double integral from t_1 to t_2 and from 0 to L; since the variations δy and δz are arbitrary, the integrand must vanish. Moreover, the terms multiplied by δy are independent of the terms multiplied by δz; consequently what seemed to be one equation may be separated into two distinct partial differential equations. Keeping in mind that the inextensibility condition has reduced the system of three variables down to two, the two equations derived describe the 3-D dynamics of a cantilevered pipe conveying fluid with additional spring constraints and an added end mass. These equations are expressed in terms of displacements v and w, and they are given below.

(i) *The y-equation*:

$$\begin{aligned}&[(m+M+m_e\delta(s-L))]\ddot{v}+EIv''''+2MU\dot{v}'+MU^2v''\\&+[m+M+m_e\delta(s-L)]gv'-v''\int_s^L g[m+M+m_e\delta(s-L)]\,\mathrm{d}s\\&+[m+M+m_e\delta(s-L)]g\left(\frac{1}{2}v'^3+\frac{1}{2}v'w'^2\right)\\&-\left(\frac{3}{2}v'^2v''+\frac{1}{2}w'^2v''+v'w'w''\right)\int_s^L g[m+M+m_e\delta(s-L)]\,\mathrm{d}s\\&+2MU\left[v'^2\dot{v}'+v'w'\dot{w}'-v''\int_s^L(v'\dot{v}'+w'\dot{w}')\,\mathrm{d}s\right]\\&+MU^2\left[v'^2v''+v'w'w''-v''\int_s^L(v'v''+w'w'')\,\mathrm{d}s\right]\\&+EI(v'^2v''''+4v'v''v'''+v''^3+v'w'w''''+3v'w''w'''+v''w'w'''+v''w''^2)\\&+v'[m+M+m_e\delta(s-L)]\int_0^s(\dot{v}'^2+v'\ddot{v}'+\dot{w}'^2+w'\ddot{w}')\,\mathrm{d}s\\&-v''\int_s^L[m+M+m_e\delta(s-L)]\int_0^s(\dot{v}'^2+v'\ddot{v}'+\dot{w}'^2+w'\ddot{w}')\,\mathrm{d}s\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+\left[(K_{yl}v + K_{ynl}v^3 + K_{yz}vw^2) + K_x v' \int_0^s (v'^2 + w'^2)\,\mathrm{d}s\right]\delta(s - L_s)\\
&- K_x v'' \mu(0 \to L_s) \int_0^{L_s} (v'^2 + w'^2)\,\mathrm{d}s = 0,
\end{aligned}
\tag{40}
$$

(ii) *The z-equation*:

$$
\begin{aligned}
&(m + M + m_e\delta(s - L))\ddot{w} + EIw'''' + 2MU\dot{w}' + MU^2 w''\\
&+ [m + M + m_e\delta(s - L)]gw' - w'' \int_s^L g[m + M + m_e\delta(s - L)]\,\mathrm{d}s\\
&+ [m + M + m_e\delta(s - L)]g\left(\frac{1}{2}w'^3 + \frac{1}{2}w'v'^2\right)\\
&- \left(\frac{3}{2}w'^2 w'' + \frac{1}{2}v'^2 w'' + w'v'v''\right) \int_s^L g[m + M + m_e\delta(s - L)]\,\mathrm{d}s\\
&+ 2MU\left[w'^2\dot{w}' + w'v'\dot{v}' - w'' \int_s^L (w'\dot{w}' + v'\dot{v}')\,\mathrm{d}s\right]\\
&+ MU^2\left[w'^2 w'' + w'v'v'' - w'' \int_s^L (w'w'' + v'v'')\,\mathrm{d}s\right]\\
&+ EI(w'^2 w'''' + 4w'w''w''' + w''^3 + w'v'v'''' + 3w'v''v''' + w''v'v''' + w''v''^2)\\
&+ w'[m + M + m_e\delta(s - L)] \int_0^s (\dot{w}'^2 + w'\ddot{w}' + \dot{v}'^2 + v'\ddot{v}')\,\mathrm{d}s\\
&- w'' \int_s^L [m + M + m_e\delta(s - L)] \int_0^s (\dot{w}'^2 + w'\ddot{w}' + \dot{v}'^2 + v'\ddot{v}')\,\mathrm{d}s\,\mathrm{d}s\\
&+ \left[(K_{zl}w + K_{znl}w^3 + K_{yz}wv^2) + K_x w' \int_0^s (v'^2 + w'^2)\,\mathrm{d}s\right]\delta(s - L_s)\\
&- K_x w'' \mu(0 \to L_s) \int_0^{L_s} (v'^2 + w'^2)\,\mathrm{d}s = 0.
\end{aligned}
\tag{41}
$$

It is noted that when analyzing the system for planar motion, say in the xy-plane, all displacements in the z-direction would be zero. Consequently, Eq. (41) would lose all of its terms, and all coupled nonlinear terms in Eq. (40) would vanish. Disregarding the intermediate springs, the equation of motion would then reduce to the familiar form found in Païdoussis and Semler (1998); or disregarding the end mass also, i.e. for just the plain cantilevered pipe, to the form found in Semler et al. (1994) or Païdoussis (1998).

All nonlinear terms are of third order. There are no second order, $\mathcal{O}(\varepsilon^2)$, terms due to the symmetrical nature of the system, i.e. any displacements in the positive y- and z-directions generate equal and opposite reaction forces for equal negative displacements; alternatively viewed, this says that replacing v by $-v$ and/or w by $-w$, one obtains the same equation.

6. Dimensionless form of the equations

Introducing nondimensional quantities to allow the analysis of not just one specific system but of a generalized system, regardless of shape and size,[3]

$$
\begin{aligned}
&\xi = \frac{s}{L}, \quad \xi_s = \frac{L_s}{L}, \quad \eta = \frac{v}{L}, \quad \zeta = \frac{w}{L}, \quad \tau = \left(\frac{EI}{m + M}\right)^{1/2} \frac{t}{L^2},\\
&u = \left(\frac{M}{EI}\right)^{1/2} UL, \quad \gamma = \frac{m + M}{EI} L^3 g, \quad \beta = \frac{M}{m + M}, \quad \Gamma = \frac{m_e}{(m + M)L},
\end{aligned}
$$

[3]In what follows, u denotes the dimensionless flow velocity, as defined in (42), for consistency with much of the published literature. This should not cause any confusion with the axial displacement u used in the derivations in Sections 2–5, which in any case has disappeared in the final dimensional Eqs. (40) and (41).

$$\kappa_x = \frac{K_x L^3}{EI}, \quad \kappa_{yl} = \frac{K_{yl} L^3}{EI}, \quad \kappa_{zl} = \frac{K_{zl} L^3}{EI},$$
$$\kappa_{ynl} = \frac{K_{ynl} L^5}{EI}, \quad \kappa_{znl} = \frac{K_{znl} L^5}{EI}, \quad \kappa_{yz} = \frac{K_{yz} L^5}{EI}, \tag{42}$$

the nondimensional equations of motion become

(i) *The y-equation*:

$$\begin{aligned}
&\eta'''' + [1 + \Gamma\delta(\xi - 1)]\ddot{\eta} + 2u\sqrt{\beta}\dot{\eta}' + u^2\eta'' \\
&+ \gamma[1 + \Gamma\delta(\xi - 1)]\eta' - \gamma\eta'' \int_\xi^1 [1 + \Gamma\delta(\xi - 1)]\,\mathrm{d}\xi \\
&+ \gamma[1 + \Gamma\delta(\xi - 1)](\tfrac{1}{2}\eta'^3 + \tfrac{1}{2}\eta'\zeta'^2) \\
&- \gamma\left(\tfrac{3}{2}\eta'^2\eta'' + \tfrac{1}{2}\zeta'^2\eta'' + \eta'\zeta'\zeta''\right) \int_\xi^1 [1 + \Gamma\delta(\xi - 1)]\,\mathrm{d}\xi \\
&+ 2u\sqrt{\beta}\left[\eta'^2\dot{\eta}' + \eta'\zeta'\dot{\zeta}' - \eta'' \int_\xi^1 (\eta'\dot{\eta}' + \zeta'\dot{\zeta}')\,\mathrm{d}\xi\right] \\
&+ u^2\left[\eta'^2\eta'' + \eta'\zeta'\zeta'' - \eta'' \int_\xi^1 (\eta'\eta'' + \zeta'\zeta'')\,\mathrm{d}\xi\right] \\
&+ [\eta'^2\eta'''' + 4\eta'\eta''\eta''' + \eta''^3 + \eta'\zeta'\zeta'''' + 3\eta'\zeta''\zeta''' + \eta''\zeta'\zeta''' + \eta''\zeta''^2] \\
&+ \eta'[1 + \Gamma\delta(\xi - 1)] \int_0^\xi (\dot{\eta}'^2 + \eta'\ddot{\eta}' + \dot{\zeta}'^2 + \zeta'\ddot{\zeta}')\,\mathrm{d}\xi \\
&- \eta'' \int_\xi^1 [1 + \Gamma\delta(\xi - 1)] \int_0^\xi (\dot{\eta}'^2 + \eta'\ddot{\eta}' + \dot{\zeta}'^2 + \zeta'\ddot{\zeta}')\,\mathrm{d}\xi\,\mathrm{d}\xi \\
&+ \left((\kappa_{yl}\eta + \kappa_{ynl}\eta^3 + \kappa_{yz}\eta\zeta^2) + \kappa_x\eta' \int_0^\xi (\zeta'^2 + \eta'^2)\,\mathrm{d}\xi\right)\delta(\xi - \xi_s) \\
&- \kappa_x\eta''\mu(0 \to \xi_s) \int_0^{\xi_s} (\zeta'^2 + \eta'^2)\,\mathrm{d}\xi = 0.
\end{aligned} \tag{43}$$

(ii) *The z-equation*:

$$\begin{aligned}
&\zeta'''' + [1 + \Gamma\delta(\xi - 1)]\ddot{\zeta} + 2u\sqrt{\beta}\dot{\zeta}' + u^2\zeta'' \\
&+ \gamma[1 + \Gamma\delta(\xi - 1)]\zeta' - \gamma\zeta'' \int_\xi^1 [1 + \Gamma\delta(\xi - 1)]\,\mathrm{d}\xi \\
&+ \gamma[1 + \Gamma\delta(\xi - 1)](\tfrac{1}{2}\zeta'^3 + \tfrac{1}{2}\zeta'\eta'^2) \\
&- \gamma(\tfrac{3}{2}\zeta'^2\zeta'' + \tfrac{1}{2}\eta'^2\zeta'' + \zeta'\eta'\eta'') \int_\xi^1 [1 + \Gamma\delta(\xi - 1)]\,\mathrm{d}\xi \\
&+ 2u\sqrt{\beta}\left[\zeta'^2\dot{\zeta}' + \zeta'\eta'\dot{\eta}' - \zeta'' \int_\xi^1 (\zeta'\dot{\zeta}' + \eta'\dot{\eta}')\,\mathrm{d}\xi\right] \\
&+ u^2\left[\zeta'^2\zeta'' + \zeta'\eta'\eta'' - \zeta'' \int_\xi^1 (\zeta'\zeta'' + \eta'\eta'')\,\mathrm{d}\xi\right] \\
&+ [\zeta'^2\zeta'''' + 4\zeta'\zeta''\zeta''' + \zeta''^3 + \zeta'\eta'\eta'''' + 3\zeta'\eta''\eta''' + \zeta''\eta'\eta''' + \zeta''\eta''^2] \\
&+ \zeta'[1 + \Gamma\delta(\xi - 1)] \int_0^\xi (\dot{\zeta}'^2 + \zeta'\ddot{\zeta}' + \dot{\eta}'^2 + \eta'\ddot{\eta}')\,\mathrm{d}\xi \\
&- \zeta'' \int_\xi^1 [1 + \Gamma\delta(\xi - 1)] \int_0^\xi (\dot{\zeta}'^2 + \zeta'\ddot{\zeta}' + \dot{\eta}'^2 + \eta'\ddot{\eta}')\,\mathrm{d}\xi\,\mathrm{d}\xi \\
&+ \left((\kappa_{zl}\zeta + \kappa_{znl}\zeta^3 + \kappa_{yz}\zeta\eta^2) + \kappa_x\zeta' \int_0^\xi (\zeta'^2 + \eta'^2)\,\mathrm{d}\xi\right)\delta(\xi - \xi_s) \\
&- \kappa_x\zeta''\mu(0 \to \xi_s) \int_0^{\xi_s} (\zeta'^2 + \eta'^2)\,\mathrm{d}\xi = 0.
\end{aligned} \tag{44}$$

By incorporating the forces associated with the end mass in the equations of motion, the boundary conditions are the same as for a linear plain cantilever beam, namely

$$\begin{aligned}&\eta(0)=\eta'(0)=\eta''(1)=\eta'''(1)=0,\\&\zeta(0)=\zeta'(0)=\zeta''(1)=\zeta'''(1)=0.\end{aligned}\tag{45}$$

7. Method of solution

The two partial differential equations have an infinite number of degrees of freedom. In order to analyze the equations numerically, the continuous system is discretized to one with a finite number of degrees of freedom. One way of doing this is to define the shape of the pipe in time through a combination of the dominant modes of a convenient subsystem, say the linear cantilever beam. Often, and for relatively low flow velocities (Païdoussis, 1998), 2–4 modes are sufficient to obtain reliable results.

Galerkin's discretization method will be applied to the equations of motion, such that

$$\eta(\xi,\tau)=\sum_{r=1}^{N}\phi_r(\xi)q_r(\tau),\tag{46}$$

$$\zeta(\xi,\tau)=\sum_{r=1}^{N}\psi_r(\xi)p_r(\tau),\tag{47}$$

where $\phi_r(\xi)$ and $\psi_r(\xi)$ are the dimensionless cantilever beam eigenfunctions, and hence appropriate comparison functions as they satisfy the same boundary conditions as the problem at hand. For the same reason, $\phi_r(\xi)=\psi_r(\xi)$, corresponding to the respective eigenfunctions in the y and z direction; however, the difference in notation will be maintained to avoid confusion in the case of the coupled terms and so that, in the end, the distinction between the η- and ζ-terms remains obvious. The corresponding generalized coordinates are $q_r(\tau)$ and $p_r(\tau)$. Once Eqs. (46) and (47) are substituted into the dimensionless equations of motion, they are multiplied by the corresponding beam eigenfunction and integrated with respect to ξ from 0 to 1. These eigenfunctions are orthonormal, i.e.

$$\int_0^1\phi_s\phi_r\,\mathrm{d}\xi=\delta_{sr},\tag{48}$$

where δ_{sr} is the Kronecker delta; it can also be shown easily that

$$\phi_r''''=\lambda_r^4\phi_r,\tag{49}$$

λ_r being the rth dimensionless eigenvalue of the cantilever beam. These relationships become useful when solving the equations of motion numerically.

The y-equation can be simplified to the following form:

$$\begin{aligned}&m_{ij}\ddot{q}_j+c_{ij}\dot{q}_j+(k_{ij}+\kappa_{ij}^{ly})q_j+(B_{ijkl}+\kappa_{ijkl}^{nly})q_jq_kq_l+D_{ijkl}q_jq_k\dot{q}_l+E_{ijkl}q_j\dot{q}_k\dot{q}_l\\&\quad+F_{ijkl}q_jq_k\ddot{q}_l+(H_{ijkl}+\kappa_{ijkl}^{yzz})q_jp_kp_l+L_{ijkl}q_jp_k\dot{p}_l+M_{ijkl}q_j\dot{p}_k\dot{p}_l+N_{ijkl}q_jp_k\ddot{p}_l=0;\end{aligned}\tag{50}$$

similarly, the z-equation becomes

$$\begin{aligned}&m_{ij}\ddot{p}_j+c_{ij}\dot{p}_j+(k_{ij}+\kappa_{ij}^{lz})p_j+(B_{ijkl}+\kappa_{ijkl}^{nlz})p_jp_kp_l+D_{ijkl}p_jp_k\dot{p}_l+E_{ijkl}p_j\dot{p}_k\dot{p}_l\\&\quad+F_{ijkl}p_jp_k\ddot{p}_l+(H_{ijkl}+\kappa_{ijkl}^{zyy})p_jq_kq_l+L_{ijkl}p_jq_k\dot{q}_l+M_{ijkl}p_j\dot{q}_k\dot{q}_l+N_{ijkl}p_jq_k\ddot{q}_l=0.\end{aligned}\tag{51}$$

The linear and nonlinear coefficients in these equations are defined as follows:

$$m_{ij}=\int_0^1\phi_i\phi_j\,\mathrm{d}\xi+\Gamma[\phi_i\phi_j]_{\xi=1},$$

$$c_{ij}=2u\sqrt{\beta}\int_0^1\phi_i\phi_j'\,\mathrm{d}\xi,$$

$$k_{ij} = \int_0^1 \phi_i \phi_j'''' \, \mathrm{d}\xi + (u^2 - \gamma \Gamma) \int_0^1 \phi_i \phi_j'' \, \mathrm{d}\xi + \gamma \int_0^1 \phi_i \phi_j' \, \mathrm{d}\xi + \gamma \Gamma [\phi_i \phi_j']_{\xi=1} - \gamma \left(\int_0^1 \phi_i \phi_j'' \, \mathrm{d}\xi - \int_0^1 \phi_i \xi \phi_j'' \, \mathrm{d}\xi \right),$$

$$B_{ijkl} = u^2 \int_0^1 \phi_i \left(\phi_j' \phi_k' \phi_l'' - \phi_j'' \int_\xi^1 \phi_k' \phi_l'' \, \mathrm{d}\xi \right) \mathrm{d}\xi + \gamma \int_0^1 \phi_i \left(\frac{1}{2} \phi_j' \phi_k' \phi_l' - \tfrac{3}{2}(1-\xi) \phi_j' \phi_k' \phi_l'' \right) \mathrm{d}\xi - \gamma \Gamma \left(\int_0^1 \phi_i \left(\frac{3}{2} \phi_j' \phi_k' \phi_l'' \right) \mathrm{d}\xi - \frac{1}{2} [\phi_i \phi_j' \phi_k' \phi_l']_{\xi=1} \right) + \int_0^1 \phi_i (\phi_j' \phi_k' \phi_l'''' + 4 \phi_j' \phi_k'' \phi_l''' + \phi_j'' \phi_k'' \phi_l'') \, \mathrm{d}\xi,$$

$$D_{ijkl} = 2u\sqrt{\beta} \int_0^1 \phi_i \left(\phi_j' \phi_k' \phi_l' - \phi_j'' \int_\xi^1 \phi_k' \phi_l' \, \mathrm{d}\xi \right) \mathrm{d}\xi,$$

$$E_{ijkl} = F_{ijkl} = \int_0^1 \phi_i \left(\phi_j' \int_0^\xi \phi_k' \phi_l' \, \mathrm{d}\xi - \phi_j'' \int_\xi^1 \int_0^\xi \phi_k' \phi_l' \, \mathrm{d}\xi \, \mathrm{d}\xi \right) \mathrm{d}\xi + \Gamma [\phi_i \phi_j']_{\xi=1} \int_0^1 \phi_k' \phi_l' \, \mathrm{d}\xi - \Gamma \int_0^1 \phi_i \phi_j'' \, \mathrm{d}\xi \int_0^1 \phi_k' \phi_l' \, \mathrm{d}\xi,$$

$$H_{ijkl} = \gamma \int_0^1 \phi_i \left[\frac{1}{2} \phi_j' \psi_k' \psi_l' - (1-\xi) \left(\frac{1}{2} \phi_j'' \psi_k' \psi_l' + \phi_j' \psi_k' \psi_l'' \right) \right] \mathrm{d}\xi - \gamma \Gamma \left(\int_0^1 \phi_i \left(\frac{1}{2} \phi_j'' \psi_k' \psi_l' + \phi_j' \psi_k' \psi_l'' \right) \mathrm{d}\xi - \frac{1}{2} [\phi_i \phi_j' \psi_k' \psi_l']_{\xi=1} \right) + u^2 \int_0^1 \phi_i \left(\phi_j' \psi_k' \psi_l'' - \phi_j'' \int_\xi^1 \psi_k' \psi_l'' \, \mathrm{d}\xi \right) \mathrm{d}\xi + \int_0^1 \phi_i (\phi_j' \psi_k' \psi_l'''' + 3 \phi_j' \psi_k'' \psi_l''' + \phi_j'' \psi_k' \psi_l''' + \phi_j'' \psi_k'' \psi_l'') \, \mathrm{d}\xi,$$

$$L_{ijkl} = 2u\sqrt{\beta} \int_0^1 \phi_i \left(\phi_j' \psi_k' \psi_l' - \phi_j'' \int_\xi^1 \psi_k' \psi_l' \, \mathrm{d}\xi \right) \mathrm{d}\xi,$$

$$M_{ijkl} = N_{ijkl} = \int_0^1 \phi_i \left(\phi_j' \int_0^\xi \psi_k' \psi_l' \, \mathrm{d}\xi - \phi_j'' \int_\xi^1 \int_0^\xi \psi_k' \psi_l' \, \mathrm{d}\xi \, \mathrm{d}\xi \right) \mathrm{d}\xi + \Gamma [\phi_i \phi_j']_{\xi=1} \int_0^1 \psi_k' \psi_l' \, \mathrm{d}\xi - \Gamma \int_0^1 \phi_i \phi_j'' \, \mathrm{d}\xi \int_0^1 \psi_k' \psi_l' \, \mathrm{d}\xi; \quad (52)$$

the spring coefficients are

$$\kappa_{ij}^{ly} = \int_0^1 \kappa_{yl} \phi_i \phi_j \delta(\xi - \xi_s) \, \mathrm{d}\xi,$$
$$\kappa_{ij}^{lz} = \int_0^1 \kappa_{zl} \psi_i \psi_j \delta(\xi - \xi_s) \, \mathrm{d}\xi,$$

$$\begin{aligned}
\kappa_{ijkl}^{nly} &= \int_0^1 (\kappa_{ynl}\phi_i\phi_j\phi_k\phi_l\delta(\xi-\xi_s)+\kappa_x \mathrm{K}_{ijkl})\,\mathrm{d}\xi,\\
\kappa_{ijkl}^{yzz} &= \int_0^1 (\kappa_{yz}\phi_i\phi_j\psi_k\psi_l\delta(\xi-\xi_s)+\kappa_x \mathrm{K}_{ijkl})\,\mathrm{d}\xi,\\
\kappa_{ijkl}^{nlz} &= \int_0^1 (\kappa_{znl}\psi_i\psi_j\psi_k\psi_l\delta(\xi-\xi_s)+\kappa_x \mathrm{K}_{ijkl})\,\mathrm{d}\xi,\\
\kappa_{ijkl}^{zyy} &= \int_0^1 (\kappa_{yz}\psi_i\psi_j\phi_k\phi_l\delta(\xi-\xi_s)+\kappa_x \mathrm{K}_{ijkl})\,\mathrm{d}\xi,
\end{aligned} \tag{53}$$

in which

$$\mathrm{K}_{ijkl} = \phi_i\phi_j'\delta(\xi-\xi_s)\int_0^{\xi}\phi_k'\phi_l'\,\mathrm{d}\xi - \mu(0\to\xi_s)\phi_i\phi_j''\int_0^{\xi_s}\phi_k'\phi_l'\,\mathrm{d}\xi,$$

or

$$\begin{aligned}
\int_0^1 \mathrm{K}_{ijkl}\,\mathrm{d}\xi &= \int_0^1\left[\phi_i\phi_j'\delta(\xi-\xi_s)\int_0^{\xi}\phi_k'\phi_l'\,\mathrm{d}\xi - \mu(0\to\xi_s)\phi_i\phi_j''\int_0^{\xi_s}\phi_k'\phi_l'\,\mathrm{d}\xi\right]\mathrm{d}\xi\\
&= [\phi_i\phi_j']_{\xi=\xi_s}\int_0^{\xi_s}\phi_k'\phi_l'\,\mathrm{d}\xi - \int_0^{\xi_s}\phi_i\phi_j''\int_0^{\xi_s}\phi_k'\phi_l'\,\mathrm{d}\xi\,\mathrm{d}\xi.
\end{aligned} \tag{54}$$

Since the eigenfunctions are the same in both y and z directions, the coefficients for the z-equation are the same as for the y-equation, as given in Eqs. (52)–(54).

8. Conclusion

In this paper, the nonlinear equations of 3-D motion of a cantilevered pipe conveying incompressible fluid have been derived to $\mathcal{O}(\varepsilon^3)$, where the lateral deflection of the pipe is considered to be of $\mathcal{O}(\varepsilon)$. In their most general form these equations account for the possible existence of (i) either a four-spring or a two-spring constraint at an axial location between the clamped and free ends of the pipe, and (ii) a dimensionally small end-mass attached to the free end of the pipe.

The equations of motion of the plain pipe (i.e. without springs and end-mass) have been derived first; they are quite similar to those obtained by Semler et al. (1994), and the derivation followed the same path. Accordingly, it was not necessary to reproduce these derivations here in too much, repetitious detail. Nevertheless, the equations obtained for motion in each of two mutually perpendicular planes contain the nonlinear cross-coupling terms associated with motion in the other direction, totally absent in the earlier derivation.

The linear equation of motion accounting for the presence of an end-mass may be found in Païdoussis and Luu (1985), and the nonlinear equation in Païdoussis and Semler (1998), in both cases for planar motions only. Nevertheless, the manner of incorporating the effect of this end-mass in the 3-D equations of motion is given in this paper in fair detail. The effect of the end-mass was incorporated in the equations of motion through a Dirac delta function, rather than in the boundary conditions, for convenience in the method of solution outlined here and used in the Part 3 (Modarres-Sadeghi et al., 2007) paper.

It should be signalled that both with and without the end-mass, an interesting aspect of the equations of motion is that they contain nonlinear inertial terms, which renders their solution rather tricky (Semler et al., 1996).

The necessary modifications to the equations of motion in order to incorporate the four- or two-spring arrays somewhere along the length of the pipe are described in considerably greater detail than the rest, as this aspect is not available, in the form treated here, elsewhere in any archival publication. It is reiterated that, although the springs are linear, they generate nonlinear terms also, because of large deformation of the pipe to which they are attached; i.e., they give rise to geometric nonlinearities. Furthermore, though at equilibrium the springs all lie in a specific plane perpendicular to the pipe, they do not stay in a plane in the course of arbitrary pipe motions, and in fact generate forces in the axial direction also.

In the main text, the springs are incorporated as if they were connected to the centreline of the pipe, which of course is only mathematically feasible. In Appendix C, the formulation is modified to correspond to the physical system in the experiments described in the Part 2 (Païdoussis et al., 2007) paper, where the springs are attached to a short thin ring fitted over the pipe at the desired location.

The equations of motion have been discretized and are, therefore, ready for solution in Part 2 (Païdoussis et al., 2007) and Part 3 (Modarres-Sadeghi et al., 2007) of this study, in which the dynamics of the system with the spring constraint and the end-mass, respectively, are examined. In addition to typical results to illustrate the dynamical behaviour of these systems, *ad hoc* experiments to test the theory are also described, and comparisons between theory and experiment are undertaken.

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Appendix A. Additional details for the four-spring configuration derivation

It seems appropriate to give more detail on the way the expressions for the forces associated with the nonlinear springs were obtained.

A.1. Expressions for R_i and n_{R_i}

As shown in Eq. (26), the resulting force from each spring may be vectorized by multiplying the deformation force of the spring, $k(R_i - L_o)$, by the unit vector along the length of the spring, $\boldsymbol{n}_{R_i}$, where

$$R_i = \sqrt{u^2 + (Q \mp v)^2 + (P \mp w)^2},$$
$$\boldsymbol{n}_{R_i} = \frac{-u\boldsymbol{i} \pm (Q \mp v)\boldsymbol{j} \pm (P \mp w)\boldsymbol{k}}{R_i}, \tag{A.1}$$

and where L_o is the unstretched length of the spring. The $\pm$ sign depends on which quadrant of the yz-plane the spring is in, at equilibrium.[4] The forces exerted by each spring, as shown in Fig. 3, are

$$\boldsymbol{F}_1 = k\left(1 - \frac{L_o}{R_1}\right)(-u\boldsymbol{i} + (Q - v)\boldsymbol{j} + (P - w)\boldsymbol{k}), \tag{A.2}$$

$$\boldsymbol{F}_2 = k\left(1 - \frac{L_o}{R_2}\right)(-u\boldsymbol{i} - (Q + v)\boldsymbol{j} + (P - w)\boldsymbol{k}), \tag{A.3}$$

$$\boldsymbol{F}_3 = k\left(1 - \frac{L_o}{R_3}\right)(-u\boldsymbol{i} - (Q + v)\boldsymbol{j} - (P + w)\boldsymbol{k}), \tag{A.4}$$

$$\boldsymbol{F}_4 = k\left(1 - \frac{L_o}{R_4}\right)(-u\boldsymbol{i} + (Q - v)\boldsymbol{j} - (P + w)\boldsymbol{k}). \tag{A.5}$$

Adding these forces and decomposing them into forces acting in the x, y and z direction, respectively, we obtain

$$F_u = -k\left[4 - L_o\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)\right]u,$$
$$F_v = -k\left[4v + L_o\left(\frac{(Q-v)}{R_1} - \frac{(Q+v)}{R_2} - \frac{(Q+v)}{R_3} + \frac{(Q-v)}{R_4}\right)\right],$$
$$F_w = -k\left[4w + L_o\left(\frac{(P-w)}{R_1} + \frac{(P-w)}{R_2} - \frac{(P+w)}{R_3} - \frac{(P+w)}{R_4}\right)\right]. \tag{A.6}$$

The inextensibility condition can be written as

$$u = -\frac{1}{2}\int_0^s (v'^2 + w'^2)\,\mathrm{d}s; \tag{A.7}$$

[4] Here, as in Sections 2–5, u stands for the axial deformation of the pipe, rather than the dimensionless flow velocity as in Section 6 and in Parts 2 and 3.

it is obvious that u is of order of magnitude $\mathcal{O}(\varepsilon^2)$, hence it is often disregarded with respect to lower-order terms in the following manipulations, for example in the expression for R_i where u^2 is of the order $\mathcal{O}(\varepsilon^4)$.

The next step consists of applying Taylor series expansions with respect to v and w, correct up to third order, $\mathcal{O}(\varepsilon^3)$, to the expressions in Eq. (A.6). After replacing P and Q by R_o and θ via

$$P = R_o \cos\theta, \quad Q = R_o \sin\theta, \quad P^2 + Q^2 = R_o^2, \tag{A.8}$$

and after some rearrangement, the expressions presented in Eqs. (27)–(29) can be obtained and are given here again for convenience

$$F_u = 2k\left(1 - \frac{L_o}{R_o}\right)\delta(s - L_s)\int_0^s (v'^2 + w'^2)\,\mathrm{d}s, \tag{A.9}$$

$$\begin{aligned} F_v = \delta(s - L_s)\Bigg\{ &- 4k\left(1 - \frac{L_o}{R_o}\cos^2\theta\right)v \\ &- 2k\frac{L_o}{R_o^3}\cos^2\theta(\cos^2\theta - 4\sin^2\theta)v^3 - 2k\frac{L_o}{R_o^3}(15\cos^2\theta\sin^2\theta - 2)vw^2\Bigg\}, \end{aligned} \tag{A.10}$$

$$\begin{aligned} F_w = \delta(s - L_s)\Bigg\{ &- 4k\left(1 - \frac{L_o}{R_o}\sin^2\theta\right)w \\ &- 2k\frac{L_o}{R_o^3}\sin^2\theta(\sin^2\theta - 4\cos^2\theta)w^3 - 2k\frac{L_o}{R_o^3}(15\cos^2\theta\sin^2\theta - 2)wv^2\Bigg\}. \end{aligned} \tag{A.11}$$

A.2. Derivation of Eq. (36) from Eqs. (33)–(35)

Some details on the steps necessary to obtain Eq. (36) from Eqs. (33)–(35) are given here to help the reader who may be interested in reproducing these equations.

Applying the inextensibility condition to the expression of virtual work acting in the x-direction, Eq. (33), we obtain

$$\begin{aligned} \delta W_u = &\int_0^L \left(K_x \int_0^s (v'^2 + w'^2)\,\mathrm{d}s\right)\delta(s - L_s)\left(-v'\delta v + \int_0^s v''\delta v\,\mathrm{d}s\right)\mathrm{d}s \\ &+ \int_0^L \left(K_x \int_0^s (v'^2 + w'^2)\,\mathrm{d}s\right)\delta(s - L_s)\left(-w'\delta w + \int_0^s w''\delta w\,\mathrm{d}s\right)\mathrm{d}s. \end{aligned} \tag{A.12}$$

Applying the property of integrals given in Eq. (24), expression (A.12) can be manipulated to obtain the following:

$$\begin{aligned} \delta W_u = &- \int_0^L \left(K_x \int_0^s (v'^2 + w'^2)\,\mathrm{d}s\right)\delta(s - L_s)w'\delta w\,\mathrm{d}s \\ &- \int_0^L \left(K_x \int_0^s (v'^2 + w'^2)\,\mathrm{d}s\right)\delta(s - L_s)v'\delta v\,\mathrm{d}s \\ &+ \int_0^L \left[K_x w'' \int_s^L \delta(s - L_s)\int_0^s (v'^2 + w'^2)\,\mathrm{d}s\,\mathrm{d}s\right]\delta w\,\mathrm{d}s \\ &+ \int_0^L \left[K_x v'' \int_s^L \delta(s - L_s)\int_0^s (v'^2 + w'^2)\,\mathrm{d}s\,\mathrm{d}s\right]\delta v\,\mathrm{d}s. \end{aligned} \tag{A.13}$$

Proceeding in a similar way for δW_v and δW_w, we obtain Eq. (36).

Appendix B. The spring forces for the two-spring configuration

It is of interest to study the dynamics of a cantilevered pipe conveying fluid constrained by an array of two intermediate springs instead of four. In this particular analysis, as shown in Fig. 4, at equilibrium the two springs are symmetrically disposed with respect to the zx-plane.

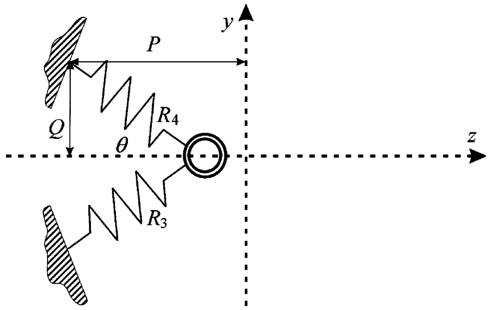

Fig. 4. Two-spring configuration, where θ , P and Q are predetermined and $P^2 + Q^2 = R_o^2$; here $L_o \neq R_o$, hence there is a pre-tension effect.

Starting from Eq. (26), with $N = 2$ for the forces exerted by the two springs acting on the pipe, and making use of $\boldsymbol{F}_3$ and $\boldsymbol{F}_4$ from Eqs. (A.5), the forces acting in the x, y and z directions, respectively, are

$$F_u = -k\left[2 - L_o\left(\frac{1}{R_3} + \frac{1}{R_4}\right)\right]u,$$
$$F_v = -k\left(2v + L_o\left(\frac{-(Q+v)}{R_3} + \frac{(Q-v)}{R_4}\right)\right),$$
$$F_w = -k\left(2(P+w) - L_o\left(\frac{(P+w)}{R_3} + \frac{(P+w)}{R_4}\right)\right). \tag{B.1}$$

Applying Taylor series expansions and the inextensibility condition, the following expressions can be obtained:

$$F_u = (K_{1-1})\delta(s - L_s)\int_0^s (v'^2 + w'^2)\,\mathrm{d}s, \tag{B.2}$$

$$F_v = (-K_{1-3}v + K_{2-2}vw + K_{3-3}v^3 + K_{3-2}vw^2)\delta(s - L_s), \tag{B.3}$$

$$F_w = (-K_0 - K_{1-2}w - K_{2-1}w^2 + \tfrac{1}{2}K_{2-2}v^2 - K_{3-1}w^3 + K_{3-2}wv^2)\delta(s - L_s), \tag{B.4}$$

where

$$K_0 = 2k(R_o - L_o)\cos\theta, \quad K_{1-1} = k\left(1 - \frac{L_o}{R_o}\right), \quad K_{1-2} = 2k\left(1 - \frac{L_o}{R_o} + \frac{L_o}{R_o}\cos^2\theta\right),$$
$$K_{1-3} = 2k\left(1 - \frac{L_o}{R_o}\cos^2\theta\right), \quad K_{2-1} = 3k\frac{L_o}{R_o^2}(\cos\theta - \cos^3\theta),$$
$$K_{2-2} = k\frac{L_o}{R_o^2}(4\cos\theta - 6\cos^3\theta), \quad K_{3-1} = k\frac{L_o}{R_o^3}(1 - 6\cos^2\theta + 5\cos^4\theta),$$
$$K_{3-2} = k\frac{L_o}{R_o^3}(2 - 15\cos^2\theta + 15\cos^4\theta), \quad K_{3-3} = k\frac{L_o}{R_o^3}(4\cos^2\theta - 5\cos^4\theta). \tag{B.5}$$

Note that θ is half of the angle separating the springs; thus, if θ is set to zero, this is equivalent to analyzing the problem with only one spring with a linear stiffness of $2k$. The constant coefficients, K_{n-m}, are used to simplify the expressions, the subscript n being an index to relate similar coefficients: $n = 1$ refers to coefficients involving $k(\ldots)$; $n = 2$ refers to those of the form $kL_o/R_o^2(\ldots)$; and $n = 3$ to those of the form $kL_o/R_o^3(\ldots)$. The subscript m is just an arbitrary index.

The virtual work associated with virtual displacements δu, δv and δw for the two-spring arrangement may be expressed as

$$\delta W_u = \int_0^L F_u \delta u\,\mathrm{d}s = \int_0^L \left(K_{1-1}\int_0^s (v'^2 + w'^2)\,\mathrm{d}s\right)\delta(s - L_s)\delta u\,\mathrm{d}s, \tag{B.6}$$

$$\delta W_v = \int_0^L F_v \delta v\,\mathrm{d}s = -\int_0^L (K_{1-3}v - K_{2-2}vw - K_{3-3}v^3 - K_{3-2}vw^2)\delta(s-L_s)\delta v\,\mathrm{d}s, \tag{B.7}$$

$$\begin{aligned}\delta W_w = \int_0^L F_w \delta w\,\mathrm{d}s = & -\int_0^L (K_0 + K_{1-2}w + K_{2-1}w^2 - \tfrac{1}{2}K_{2-2}v^2)\delta(s-L_s)\,\delta w\,\mathrm{d}s \\ & -\int_0^L (K_{3-1}w^3 - K_{3-2}wv^2)\delta(s-L_s)\,\delta w\,\mathrm{d}s.\end{aligned} \tag{B.8}$$

By applying the inextensibility condition and utilizing Eq. (24) in the expression of virtual work acting in the axial direction, δW_u, and after some further manipulation, one obtains the following expression for the virtual work done by a two-spring arrangement symmetric in the zx-plane and attached to the pipe at $s = L_s$:

$$\begin{aligned}&\int_{t_1}^{t_2} \delta W\,\mathrm{d}t \\ &= \int_{t_1}^{t_2} (\delta W_u + \delta W_v + \delta W_w)\,\mathrm{d}t \\ &- \int_{t_1}^{t_2}\int_0^L (K_{1-3}v - K_{2-2}vw - K_{3-3}v^3 - K_{3-2}vw^2)\delta(s-L_s)\,\delta v\,\mathrm{d}s\,\mathrm{d}t \\ &- \int_{t_1}^{t_2}\int_0^L K_{1-1}\left[v'\delta(s-L_s)\int_0^s (v'^2 + w'^2)\,\mathrm{d}s - v''\mu(0\to L_s)\int_0^{L_s}(v'^2+w'^2)\,\mathrm{d}s\right]\delta v\,\mathrm{d}s\,\mathrm{d}t \\ &- \int_{t_1}^{t_2}\int_0^L \left(K_0 + K_{1-2}w + K_{2-1}w^2 - \frac{1}{2}K_{2-2}v^2 + K_{3-1}w^3 - K_{3-2}wv^2\right)\delta(s-L_s)\,\delta w\,\mathrm{d}s\,\mathrm{d}t \\ &- \int_{t_1}^{t_2}\int_0^L K_{1-1}\left[w'\delta(s-L_s)\int_0^s (v'^2 + w'^2)\,\mathrm{d}s - w''\mu(0\to L_s)\int_0^{L_s}(v'^2+w'^2)\,\mathrm{d}s\right]\delta w\,\mathrm{d}s\,\mathrm{d}t.\end{aligned} \tag{B.9}$$

This expression can be used to replace the four-spring array in the dimensionless equations of motion; specifically, the last three terms in each equation are replaced by the following:

y-equation:

$$\begin{aligned}&+(\kappa_{1-3}\eta - \kappa_{2-2}\zeta\eta - \kappa_{3-3}\eta^3 - \kappa_{3-2}\eta\zeta^2)\delta(\xi-\xi_s) \\ &\quad + \kappa_{1-1}\left(\eta'\delta(\xi-\xi_s)\int_0^\xi (\zeta'^2+\eta'^2)\,\mathrm{d}\xi - \eta''\mu(0\to\xi_s)\int_0^{\xi_s}(\zeta'^2+\eta'^2)\,\mathrm{d}\xi\right),\end{aligned} \tag{B.10}$$

z-equation:

$$\begin{aligned}&+(\kappa_0 + \kappa_{1-2}\zeta + \kappa_{2-1}\zeta^2 - \frac{1}{2}\kappa_{2-2}\eta^2 + \kappa_{3-1}\zeta^3 - \kappa_{3-2}\zeta\eta^2)\delta(\xi-\xi_s) \\ &\quad + \kappa_{1-1}\left(\zeta'\delta(\xi-\xi_s)\int_0^\xi (\zeta'^2+\eta'^2)\,\mathrm{d}\xi - \zeta''\mu(0\to\xi_s)\int_0^{\xi_s}(\zeta'^2+\eta'^2)\,\mathrm{d}\xi\right),\end{aligned} \tag{B.11}$$

where

$$\kappa_0 = \frac{K_0L^2}{EI}, \kappa_{1-i} = \frac{K_{1-i}L^3}{EI}, \quad \kappa_{2-i} = \frac{K_{2-i}L^4}{EI}, \quad \kappa_{3-i} = \frac{K_{3-i}L^5}{EI}. \tag{B.12}$$

An interesting observation can be made here: second-order, $\mathcal{O}(\varepsilon^2)$, terms are present in this case because of the asymmetrical nature of the array of springs. Also, if the initial quantities L_o and R_o are not equal, i.e. if the springs are fixed to the pipe in a pre-tensioned (extended) position, the initial position of the pipe, when $U = 0$, will not be along the x-axis but rather in a slightly bent state of equilibrium, as cross sectionally shown in Fig. 4.

Appendix C. Moments caused by physical attachment of the springs

With certain parameters, notably when the spring position along the length of the pipe is close to the free end, the experimentally observed divergence (buckling) was found to occur initially in a plane perpendicular to the theoretically predicted plane of divergence (Saaid, 1999; Wadham-Gagnon, 2004). This behaviour seems to be related to the assumption that the springs are attached to the centreline of the pipe. In the experimental set-up, these springs are attached to a thin ring, which in turn is mounted on the outside of the pipe. Moments due to this configuration were previously thought to be negligible, but they turn out to have a first order, $\mathcal{O}(\varepsilon)$, effect in the equations of motion; therefore, they may influence the dynamics, and more specifically the plane of least resistance in which the system will initially diverge. An adequate model for these moments is presented here.

C.1. Set-up of the mathematical model

Each spring is assigned a vector, $\boldsymbol{a}_i = a_{ix}\boldsymbol{i} + a_{iy}\boldsymbol{j} + a_{iz}\boldsymbol{k}$, where subscripts $i = 1, 2, 3, 4$, refer to each of the springs. Let vector $\boldsymbol{a}_i$ be normal to the centreline of the pipe passing through the point of attachment of each spring, such that

$$\boldsymbol{\tau}_{L_S} \cdot \boldsymbol{a}_i = 0, \tag{C.1}$$

where $\boldsymbol{\tau}_{L_S}$ is a unit vector tangent to the pipe centreline at $s = L_s$, which can be expressed as

$$\boldsymbol{\tau}_{L_s} = [x'\boldsymbol{i} + y'\boldsymbol{j} + z'\boldsymbol{k}]\delta(s - L_s). \tag{C.2}$$

It is quite obvious that, when the pipe is at rest in the equilibrium configuration, the x-component of vector $\boldsymbol{a}_i$, a_{ix}, is zero since the springs are in the yz-plane. Assuming there is no twist in the pipe, the y- and z-components of $\boldsymbol{a}_i$ can be approximated as known constants determined by their respective initial spring configurations. From Fig. 5, these components are expressed as follows:

$$a_{iy} = \pm \sin\psi, \quad a_{iz} = \pm \cos\psi. \tag{C.3}$$

Solving for a_{ix} by substituting (C.2) into (C.1),

$$a_{ix} = \left(\frac{-y'a_{iy} - z'a_{iz}}{x'}\right)\delta(s - L_s),$$

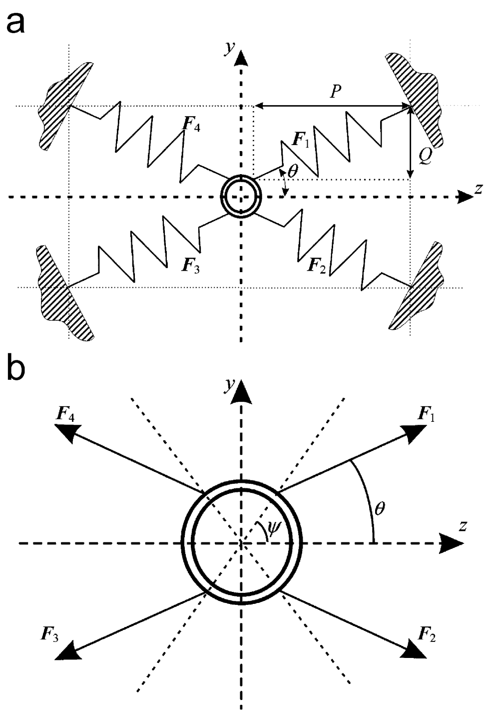

Fig. 5. (a) Display of a four spring array, when the springs are assumed to be connected to points on the outer surface of the pipe. (b) Expanded view for the attachment of the springs on to the surface of the pipe, corresponding to the configuration used in the experiments, showing that the line of action of the springs does not necessarily go through the centreline of the pipe.

from which a nonunit vector, say $\boldsymbol{a}_i'$ for the sake of the argument, can be obtained

$$\boldsymbol{a}_i' = \left(\frac{-y' a_{iy} - z' a_{ix}}{x'}\right)\delta(s - L_s)\boldsymbol{i} \pm \sin\psi\boldsymbol{j} \pm \cos\psi\boldsymbol{k};$$

using linear Taylor series expansions,

$$\frac{1}{x'} \simeq 1 + \frac{1}{2}y'^2 + \frac{1}{2}z'^2 \simeq 1,$$

and

$$\frac{1}{\sqrt{(y' a_{iy} + z' a_{iz})^2 + 1}} \simeq 1 - (y' a_{iy})^2 - (y' a_{iy} z' a_{iz})^2 - (z' a_{iz})^2 \simeq 1.$$

It is readily seen that a unit vector $\boldsymbol{a}_i$ can be obtained from

$$\boldsymbol{a}_i = \frac{\boldsymbol{a}_i'}{\|\boldsymbol{a}_i'\|} = (\mp y' \sin\psi \mp z' \cos\psi)\delta(s - L_s)\boldsymbol{i} \pm \sin\psi\boldsymbol{j} \pm \cos\psi\boldsymbol{k}. \tag{C.4}$$

Using the parallel axes theorem, the spring force acting on the pipe may be represented as a force–moment couple acting on the pipe centreline, where the force remains the same and the moment is expressed as

$$\boldsymbol{M}_i = \alpha\boldsymbol{a}_i \times \boldsymbol{F}_i, \tag{C.5}$$

where $\boldsymbol{F}_i$ is the force exerted by spring i, and coefficient α is the distance between the centreline of the pipe at $s = L_s$ and the point of connection of each spring. In this case, we take α to have the same value for each spring. The $\boldsymbol{F}_i$ forces are the same as those given in Eqs. (A.2)–(A.5).

Finally, the total moment, $\boldsymbol{M}$, is obtained by summing the $\boldsymbol{M}_i$s of each spring

$$\boldsymbol{M} = \sum_{i=1}^{4} \boldsymbol{M}_i = M_y\boldsymbol{j} + M_z\boldsymbol{k}. \tag{C.6}$$

Summing the $\boldsymbol{M}_i$'s obtained from Eq. (C.5) with Eq. (C.4) and forces in Eqs. (A.2)–(A.5), and after substituting P and Q by R_o and θ via

$$P = R_o\cos\theta, \quad Q = R_o\sin\theta, \quad P^2 + Q^2 = R_o^2, \tag{C.7}$$

the vector components of $\boldsymbol{M}$ become

$$M_y = 4\alpha k\cos(\psi)\cos(\theta)(R_o - L_o)z', \tag{C.8}$$

$$M_x = -4\alpha k\sin(\psi)\sin(\theta)(R_o - L_o)y', \tag{C.9}$$

C.2. The virtual work expression

The virtual work done by moment $\boldsymbol{M}$ on the system may be expressed as

$$\delta W_M = \boldsymbol{M} \cdot \delta\boldsymbol{\Theta}, \tag{C.10}$$

where $\boldsymbol{\Theta} = \Theta_y\boldsymbol{j} + \Theta_z\boldsymbol{k}$ is the pipe rotation vector; note that the assumption of no axial-twist eliminates rotations in the x-direction.

Expressing virtual rotation $\delta\Theta_y$ in terms of Eulerian coordinates and keeping first-order, $\mathcal{O}(\varepsilon)$, terms only, the following can be obtained from the centreline tangent vector:

$$\delta\Theta_y \simeq \delta(\tan\Theta_y) = \delta\left(\frac{z'}{x'}\right) \simeq \delta z', \tag{C.11}$$

similarly for virtual rotation $\delta\Theta_z$,

$$\delta\Theta_z \simeq \delta(\tan\Theta_z) = \delta\left(\frac{y'}{x'}\right) \simeq \delta y'. \tag{C.12}$$

Putting together what has been given so far in this appendix, the moments due to the spring configuration will contribute to the dynamics of the system as stated in Hamilton's principle, see expression (4), such that

$$\begin{aligned}\int_{t_1}^{t_2} \delta W_M \,\mathrm{d}t &= \int_{t_1}^{t_2} (\boldsymbol{M} \cdot \delta\boldsymbol{q})\,\mathrm{d}t = \int_{t_1}^{t_2} (M_y \delta\Theta_y + M_x \delta\Theta_z)\,\mathrm{d}t \\ &= \int_{t_1}^{t_2} (M_y \delta z' + M_z \delta y')\,\mathrm{d}t = \int_{t_1}^{t_2} (M_y \delta w' + M_z \delta v')\,\mathrm{d}t. \end{aligned} \tag{C.13}$$

Substituting Eqs. (C.8) and (C.9) into Eq. (C.13) and integrating by parts, we obtain

$$\begin{aligned}\int_{t_1}^{t_2} \delta W_M \,\mathrm{d}t &= \int_{t_1}^{t_2} (\boldsymbol{M} \cdot \delta\boldsymbol{q})\,\mathrm{d}t \\ &= \int_{t_1}^{t_2}\int_0^L [(4\alpha k \cos(\psi)\cos(\theta)(R_o - L_o)w')\,\delta w']\delta(s - L_s)\,\mathrm{d}s\,\mathrm{d}t \\ &\quad - \int_{t_1}^{t_2}\int_0^L [(4\alpha k \sin(\psi)\sin(\theta)(R_o - L_o)v')\,\delta v']\delta(s - L_s)\,\mathrm{d}s\,\mathrm{d}t \\ &= 4\alpha k \cos(\psi)\cos(\theta)(R_o - L_o)\int_{t_1}^{t_2}\left\{[w'\delta(s - L_s)\,\delta w]_0^L - \int_0^L (w''\delta w)\delta(s - L_s)\,\mathrm{d}s\right\}\mathrm{d}t \\ &\quad - 4\alpha k \sin(\psi)\sin(\theta)(R_o - L_o)\int_{t_1}^{t_2}\left\{[v'\delta(s - L_s)\,\delta v]_0^L - \int_0^L (v''\delta v')\delta(s - L_s)\,\mathrm{d}s\right\}\mathrm{d}t; \end{aligned} \tag{C.14}$$

the linear contribution of the moments induced by the springs may be expressed in the form of virtual work as follows:

$$\begin{aligned}\int_{t_1}^{t_2} \delta W_M \,\mathrm{d}t = &- 4\alpha k \cos(\psi)\cos(\theta)(R_o - L_o)\int_{t_1}^{t_2}\int_0^L (w''\delta w)\delta(s - L_s)\,\mathrm{d}s\,\mathrm{d}t \\ &+ 4\alpha k \sin(\psi)\sin(\theta)(R_o - L_o)\int_{t_1}^{t_2}\int_0^L (v''\delta v)\delta(s - L_s)\,\mathrm{d}s\,\mathrm{d}t. \end{aligned} \tag{C.15}$$

Transforming Eq. (C.15) into dimensionless form

$$\int_{\tau_1}^{\tau_2} \delta W_M \,\mathrm{d}\tau = -\int_{\tau_1}^{\tau_2}\int_0^1 \Pi_y \zeta'' \delta(\xi - \xi_s)\delta\zeta\,\mathrm{d}\xi\,\mathrm{d}\tau + \int_{\tau_1}^{\tau_2}\int_0^1 \Pi_z \eta'' \delta(\xi - \xi_s)\delta\eta\,\mathrm{d}\xi\,\mathrm{d}\tau, \tag{C.16}$$

where

$$\Pi_y = 4\alpha k \cos(\psi)\cos(\theta)(R_o - L_o)\frac{L}{EI}, \quad \Pi_z = 4\alpha k \sin(\psi)\sin(\theta)(R_o - L_o)\frac{L}{EI}. \tag{C.17}$$

References

Bajaj, A.K., Sethna, P.R., 1984. Flow induced bifurcations to three-dimensional oscillatory motions in continuous tubes. SIAM Journal of Applied Mathematics 44, 270–286.

Benjamin, T.B., 1961a. Dynamics of a system of articulated pipes conveying fluid, I. Theory. Proceedings of the Royal Society A261, 457–486.

Benjamin, T.B., 1961b. Dynamics of a system of articulated pipes conveying fluid, II. Experiments. Proceedings of the Royal Society A261, 487–499.

Copeland, G.S., Moon, F.C., 1992. Chaotic flow-induced vibration of a flexible tube with end mass. Journal of Fluids and Structures 6, 705–718.

Feodos'ev, V.P., 1951. Vibration and stability of a pipe when liquid flows through it. Inzhenernyi Sbornik 10, 169–170.

Gregory, R.W., Païdoussis, M.P., 1966a. Unstable oscillations of tubular cantilevers conveying fluid—I. Theory. Proceedings of the Royal Society A293, 512–527.

Gregory, R.W., Païdoussis, M.P., 1966b. Unstable oscillations of tubular cantilevers conveying fluid—II. Experiments. Proceedings of the Royal Society A293, 528–542.

Housner, G.W., 1952. Bending vibrations of a pipe line containing flowing fluid. Journal of Applied Mechanics 19, 205–208.

Lundgren, T.S., Sethna, P.R., Bajaj, A.K., 1979. Stability boundaries for flow induced motions of tubes with an inclined terminal nozzle. Journal of Sound and Vibration 64, 553–571.

Modarres-Sadeghi, Y., Semler, C., Wadham-Gagnon, M., Païdoussis, M.P., 2007. Dynamics of cantilevered pipes conveying fluid. Part 3: three-dimensional dynamics in the presence of an end-mass. Journal of Fluids and Structures 23, 589–603.

Païdoussis, M.P., 1993. The 1992 Calvin Rice Lecture: some curiosity-driven research in fluid–structure interactions and its current applications. ASME Journal of Pressure Vessel Technology 115, 2–14.

Païdoussis, M.P., 1998. Fluid–Structure Interactions: Slender Structures and Axial Flow. Academic Press, London.

Païdoussis, M.P., Li, G.X., 1993. Pipes conveying fluid: a model dynamical problem. Journal of Fluids and Structures 7, 137–204.

Païdoussis, M.P., Luu, T.P., 1985. Dynamics of a pipe aspirating fluid such as might be used in ocean mining. ASME Journal of Energy Resources Technology 107, 250–255.

Païdoussis, M.P., Semler, C., 1993. Nonlinear dynamics of a fluid-conveying cantilevered pipe with an intermediate spring support. Journal of Fluids and Structures 7, 269–298.

Païdoussis, M.P., Semler, C., 1998. Non-linear dynamics of a fluid-conveying cantilevered pipe with a small mass attached at the free end. Journal of Non-Linear Mechanics 33, 15–32.

Païdoussis, M.P., Semler, C., Wadham-Gagnon, M., Saaid, S., 2007. Dynamics of cantilevered pipes conveying fluid. Part 2: dynamics of the system with intermediate spring support. Journal of Fluids and Structures 23, xxx–xxx.

Saaid, S., 1999. Nonlinear dynamics of a fluid-conveying cantilevered pipe with an intermediate nonlinear spring support. B.Eng. Honours Thesis, Department of Mechanical Engineering, McGill University, Montreal, Québec, Canada.

Semler, C., Li, G.X., Païdoussis, M.P., 1994. The non-linear equations of motion of pipes conveying fluid. Journal of Sound and Vibration 169, 577–599.

Semler, C., Gentleman, W.C., Païdoussis, M.P., 1996. Numerical solutions of second order implicit non-linear ordinary differential equations. Journal of Sound and Vibration 195, 553–574.

Steindl, A., Troger, H., 1988. Flow induced bifurcations to 3-dimensional motions of tubes with an elastic support. In: Besserling, J.F., Eckhaus, W. (Eds.), Trends in Applications of Mathematics to Mechanics. Springer, Berlin, pp. 128–138.

Steindl, A., Troger, H., 1991. Non-Linear Stability and Bifurcation Theory. Springer, Berlin.

Steindl, A., Troger, H., 1995. Nonlinear three-dimensional oscillations of elastically constrained fluid conveying viscoelastic tubes with perfect broken $\mathcal{O}(2)$-symmetry. Nonlinear Dynamics 7, 165–193.

Steindl, A., Troger, H., 1996. Heteroclinic cycles in the three-dimensional post-bifurcation motion of $\mathcal{O}(2)$-symmetrical fluid conveying tubes. Applied Mathematics and Computation 78, 269–277.

Wadham-Gagnon, M., 2004. Three-dimensional nonlinear dynamics of cantilevered pipes conveying fluid with nonlinear constraints. B. Eng. Honours Thesis, Department of Mechanical Engineering, McGill University, Montreal, Québec, Canada.